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European Journal
of Combinatorics

European Journal of Combinatorics 24 (2003) 557–572

www.elsevier.com/locate/ejc

Quasi-median graphs, their generalizations, and tree-like equalities

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Received 5 October 2002; accepted 20 February 2003

Abstract

Three characterizations of quasi-median graphs are proved, for instance, they are partial Hamming graphs without convex house and convex Q_3^- such that certain relations on their edge sets coincide. Expansion procedures, weakly 2-convexity, and several relations on the edge set of a graph are essential for these results. Quasi-semimedial graphs are characterized which yields an additional characterization of quasi-median graphs. Two equalities for quasi-median graphs are proved. One of them asserts that if α_i , $i \geq 0$, denotes the number of induced Hamming subgraphs of a quasi-median graph, then $\sum_{i \geq 0} (-1)^i \alpha_i = 1$. Finally, an Euler-type formula is derived for graphs that can be obtained by a sequence of connected expansions from K_1 .

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MSC (2000): 05C75; 05C12; 05C05

Keywords: Quasi-median graph; Partial Hamming graph; Quasi-semimedial graph; Expansion procedure; Tree-like equality; Euler-type formula

1. Introduction

Median and quasi-median graphs are well studied classes of graphs, cf. [1, 7, 12, 17, 19–21, 25, 32]. Quasi-median graphs have been introduced by Mulder [25] as a natural nonbipartite extension of median graphs. Chung et al. [12] and independently Wilkeit [32] proved that they are the weak retracts of Hamming graphs. On the other hand, Hamming graphs are the regular quasi-median graphs [25]. Chastand [6] extended the

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above retraction result to infinite graphs. In [1] a survey of characterizations of quasi-median graphs is given including some new ones.

Quasi-median graphs have an interesting application in location theory. Namely, they are precisely the graphs for which a certain dynamic location problem provides a finite solution, see [11, 12] or [19] for more details. From the algorithmic point of view it is an easy observation that quasi-median graphs can be recognized in polynomial time. Feder's general approach of [14] yields an $O(mn)$ algorithm, where m is the number of edges and n the number of vertices of a given graph. The fastest known recognition algorithm is due to Hagauer [17] and is of complexity $O(M(m, n) + m \log n)$, where $M(m, n)$ denotes the complexity of recognizing median graphs. (Currently $M(m, n) = O(n^{1.41}(\log n)^{2.82})$, see [19].)

Partial cubes, that is, isometric subgraphs of hypercubes, were first investigated by Graham and Pollak [15], see also [10, 13, 33]. Nonbipartite extensions of this class are isometric subgraphs of Hamming graphs, called partial Hamming graphs, see [8, 16, 31]. Since (weak) retracts are isometric subgraphs, quasi-median graphs are partial Hamming graphs. In addition, quasi-median graphs are also quasi-semimedian graphs, the class of graphs that forms a nonbipartite extension of semimedian graphs introduced in [18].

In this paper we consider the quasi-median graphs and their generalizations: weakly modular graphs, partial Hamming graphs, quasi-semimedian graphs, and graphs that can be obtained from K_1 by connected expansions.

In the next section we introduce necessary concepts and recall some known results. We follow with a section in which quasi-median graphs are introduced and relevant characterizations are given. Quasi-semimedian graphs are also presented and a result of [18] is extended from semimedian to quasi-semimedian graphs. We continue with a section containing three characterizations of quasi-median graphs. We show that quasi-median graphs are precisely partial Hamming graphs which include no convex house or Q_3^- , and for which certain relations on their edge sets coincide. We also prove that quasi-median graphs can be described as quasi-semimedian graphs which contain no convex house or Q_3^- . In Section 5 we study quasi-semimedian graphs, in particular we give their characterization. This in turn enables us to obtain another characterization of quasi-median graphs. In the last section we first prove that for a quasi-median graph G the following holds:

$$\sum_{i \geq 0} (-1)^i \alpha_i = 1 \quad \text{and} \quad -t = \sum_{i \geq 0} (-1)^i i \alpha_i.$$

Here t is a dimension of G and α_i the number of induced Hamming subgraphs of G of degree i . These results generalize such equalities for median graphs [27]. We conclude the paper by proving that for a graph G that can be obtained by a sequence of connected expansions from K_1 , $2n - m - k \leq 2$ holds, where we have equality if and only if G is $C_t \square K_2$ -free ($t \geq 3$) and K_4 -free. This result extends all such previously known Euler-type formulae.

2. Preliminaries

The *interval* $I(u, v)$ between vertices u, v of a connected graph G is the set of vertices of all shortest paths between u and v in G . A graph G is a *median graph* if $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$ for all triples of vertices u, v, w of G .

A graph G satisfies the *triangle property* if for any vertices $u, x, y \in V(G)$ where $d(u, x) = d(u, y) = k \geq 2$ such that $xy \in E(G)$, there exists a common neighbour v of x and y with $d(u, v) = k - 1$. A graph G satisfies the *quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$ with z a common neighbour of x and y , there exists a common neighbour v of x and y such that $d(u, v) = d(u, x) - 1$. A graph which fulfils the quadrangle property and the triangle property is called *weakly modular*.

A subgraph H of a graph G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the length of a shortest path in G from u to v . A connected subgraph H of G is called *convex* if for every two vertices from H all shortest paths are contained in H . It is easy to see that the intersection of two convex subgraphs is also convex. A *convex closure* of a subgraph H of G is defined as the smallest convex subgraph of G which contains H . A subgraph H of a graph G is called *gated* in G if for every $x \in V(G)$ there exists a vertex u in H such that $u \in I(x, v)$ for all $v \in V(H)$. Note that if for some x such a vertex u in $V(H)$ exists, it must be unique.

An induced connected subgraph H of a graph G is *2-convex* if for any two vertices u and v of H with $d_G(u, v) = 2$, every common neighbour of u and v belongs to H . We call an induced subgraph H of a graph G *weakly 2-convex* if for any two vertices $u, v \in V(H)$ with $d_H(u, v) = 2$, every common neighbour of u and v belongs to H . The path on five vertices is a weakly 2-convex but not 2-convex subgraph of C_6 . Chepoi [9] and Bandelt and Chepoi [2, Lemma 1] observed that a connected subgraph of a weakly modular graph is weakly 2-convex if and only if it is convex. In addition, a convex subgraph is gated if and only if it is triangle-closed, where a subgraph H of a graph G is *triangle-closed* if H contains a triangle as soon it contains one of its edges. For further reference we thus state:

Lemma 2.1. *Let G be a weakly modular graph. For an induced subgraph H of G the following assertions are equivalent:*

- (i) H is gated.
- (ii) H is convex and triangle-closed.
- (iii) H is connected, triangle-closed, and weakly 2-convex.

The equivalence between (i) and (ii) has also been noticed in [1, Lemma 2]. It is easy to see that an isometric subgraph is weakly 2-convex if and only if it is 2-convex. Therefore, we can also deduce a result of Vesel [30] which claims that subgraphs of pseudo-median graphs are gated precisely when they are 2-convex, triangle-closed, and isometric.

The *Cartesian product* $G = G_1 \square G_2 \square \cdots \square G_k$ of graphs G_1, G_2, \dots, G_k has vertices $V(G) = V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ and vertices $u = (u_1, \dots, u_k)$, $v = (v_1, \dots, v_k)$ of G are adjacent if there exists an index j ($1 \leq j \leq k$) such that $u_j v_j \in E(G_j)$ and $u_i = v_i$ for all $i \in \{1, 2, \dots, k\} \setminus \{j\}$. If all the factors in a Cartesian product are complete graphs then G is called a *Hamming graph* and in particular if all k factors are K_2

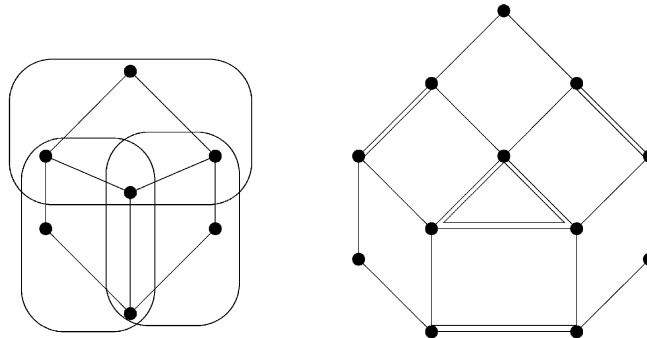


Fig. 1. Connected expansion.

then G is a *hypercube* denoted Q_k . Isometric subgraphs of hypercubes are called *partial cubes* and isometric subgraphs of Hamming graphs are *partial Hamming graphs*.

Next we introduce several relations defined on the edge set of a graph G that are essential for our investigations. For an edge ab of a graph G let $W_{ab} = \{x \in V(G) : d(x, a) < d(x, b)\}$. Then Djoković's relation \sim is defined as follows [13]: edges xy , $ab \in E(G)$ are in relation \sim if $x \in W_{ab}$ and $y \in W_{ba}$. The relation is reflexive and symmetric but it is in general not transitive, cf. $K_{2,3}$. It is well known that \sim is a transitive relation for partial Hamming graphs (see [31]).

A relation \approx was introduced in [3] (denoted there by Δ) on the edge set of a connected graph as follows. Edges e, f are in relation \approx , if $e \sim f$ or there exist edges $e', f' \in E(G)$ of the same clique, such that $e \sim e'$ and $f \sim f'$. (Note the meaning of our notation: \approx is used because, roughly speaking, we extend the relation \sim by double applications of it over cliques.) Obviously, \approx is reflexive, symmetric, and $\sim \subseteq \approx$. The relation \approx is transitive for partial Hamming graphs [3]. It is illustrated in Fig. 1, where we infer that the marked edges, obtained in an expansion step, form an equivalence class of this relation.

Edges e and f are in relation δ if $e = f$ or e and f are opposite edges of an induced square in G . (By a square we mean a 4-cycle.) We say that edges e and f are in relation κ if e and f belong to a common complete subgraph of G .

Finally, a graph obtained from $K_2 \square K_3$ by deletion of a vertex is called a *house*, Q_3^- denotes the 3-cube minus a vertex, $K_4 - e$ is the complete graph on four vertices minus an edge, and $\langle X \rangle$ stands for the subgraph induced by the vertex set X .

3. Quasi-(semi)median graphs

Recall that for an edge ab of a graph G , $W_{ab} = \{x \in V(G) : d(x, a) < d(x, b)\}$. In addition let

$$U_{ab} = \{x \in W_{ab} : x \text{ has a neighbour } y \text{ in } W_{ba}\}.$$

A graph is *quasi-median* if every clique (that is, a maximal complete subgraph) in a graph is gated and for any edge ab , U_{ab} is convex. We will need the following characterization of quasi-median graphs due to Chung et al. [12].

Theorem 3.1 ([12]). *A graph G is quasi-median if and only if G is weakly modular and does not contain $K_4 - e$ or $K_{2,3}$ as an induced subgraph.*

Semimedial graphs were introduced in [18] as partial cubes for which every set U_{ab} is connected. A natural nonbipartite extension of semimedial graphs are *quasi-semimedial graphs* introduced as partial Hamming graphs for which every set U_{ab} is connected [3]. Note that in [3] these graphs were called semi-quasi-median since they lie between partial Hamming graphs and quasi-median graphs, just as semimedial graphs lie between partial cubes and median graphs. Clearly, bipartite quasi-semimedial graphs are precisely semimedial graphs which is reflected in their new name—quasi-semimedial graphs.

It was shown in [18] that a bipartite graph is a semimedial graph if and only if $\delta^* = \sim$. This result can be extended to quasi-semimedial graphs as follows.

Proposition 3.2. *A graph is quasi-semimedial if and only if it is a partial Hamming graph with $\delta^* = \sim$.*

Proof. Let G be quasi-semimedial. Since in partial Hamming graphs \sim is transitive, and we always have $\delta \subseteq \sim$, it follows that $\delta^* \subseteq \sim$. On the other hand, if $ab \sim uv$ for $ab, uv \in E(G)$ then $u \in U_{ab}$, and since U_{ab} is connected there exists a path from u to a which lies entirely in U_{ab} . We now easily deduce that $ab\delta^*uv$.

Conversely, let $\delta^* = \sim$ and suppose that U_{ab} is not connected for $ab \in E(G)$. Then there exists an edge uv in relation \sim with ab such that any path in W_{ab} between u and a has at least one vertex in $W_{ab} \setminus U_{ab}$. We claim that then uv is not in relation δ^* with ab . Indeed, if uv were in relation δ^* with ab , then the vertices of one side of edges which are in relation δ^* with ab would induce a path in U_{ab} between u and a . \square

Odd cycles are examples of graphs for which $\delta^* = \sim$ holds. Indeed, both relations are trivial. As odd cycles of length at least 5 are not partial Hamming graphs (on the other hand, they can be embedded as induced subgraphs into Hamming graphs), we must assume in the above proposition that G is a partial Hamming graph.

Let us present a class of quasi-semimedial graphs that are not quasi-median. Take the Cartesian product of k paths, and select a set of k -cubes such that for any two k -cubes their edges are from different \sim equivalence classes. Then to each k -cube of this set add all possible edges between its vertices, that is, each Q_k is transformed into K_{2^k} . Note that the resulting graph is not quasi-median (unless the product of paths is in some sense trivial), but it is a partial Hamming graph which can be derived from the definitions of both classes (alternatively, one can use an expansion procedure described below to see that they are partial Hamming graphs). By Proposition 3.2 this partial Hamming graph is quasi-semimedial.

The notion of expansion was first introduced by Mulder in [24]; all other notions of expansion were derived from this. For our purposes, we recall the following general expansion, introduced by Chepoi [8] in the following way.

Definition 3.3. Let G be a connected graph and let W_1, W_2, \dots, W_k be subsets of $V(G)$ such that:

- (1) $W_i \cap W_j \neq \emptyset$ for all $i, j \in \{1, 2, \dots, k\}$;
- (2) $\cup_{i=1}^k W_i = V(G)$;
- (3) there are no edges between sets $W_i \setminus W_j$ and $W_j \setminus W_i$ for all $i, j \in \{1, 2, \dots, k\}$;
- (4) subgraphs $\langle W_i \rangle, \langle W_i \cup W_j \rangle$ are isometric in G for all $i, j = 1, 2, \dots, k$.

Then to each vertex $x \in V(G)$ we associate a set $\{i_1, i_2, \dots, i_t\}$ of all indices i_j , where $x \in W_{i_j}$. A graph G^* is called an *expansion of G relative to the sets W_1, W_2, \dots, W_k* if it is obtained from G in the following way:

- (5) replace each vertex x of G with a clique with vertices $x_{i_1}, x_{i_2}, \dots, x_{i_t}$;
- (6) if an index i_s belongs to both sets $\{i_1, i_2, \dots, i_t\}, \{i'_1, i'_2, \dots, i'_l\}$ corresponding to adjacent vertices x and y in G then let $x_{i_s} y_{i_s} \in E(G^*)$.

Moreover, by imposing extra conditions to the above definition, we obtain some special expansions. If $W_i \cap W_j$ induce connected subgraphs, then this is called a *connected expansion*. If, in addition, $W_i \cap W_j = U$ for all $i, j = 1, 2, \dots, k$ where $\langle U \rangle$ is a gated subgraph in G , and all subgraphs $\langle W_i \rangle$ are also gated, then this is called a *gated expansion*. If the number k of subsets involved in the expansion equals 2, then the expansion is called *binary*. An example of a (connected) expansion is given on Fig. 1.

The following theorem collects expansion theorems that are of interest to us. The first result is due to Chepoi [8], the second to Mulder [25], cf. also Bandelt et al. [1], while the last one is given in [18] for the bipartite case and extended in [3] to the general case.

Theorem 3.4. *Let G be a graph.*

- (i) G is a partial Hamming graph if and only if it can be obtained from K_1 by a sequence of expansions.
- (ii) G is a quasi-median graph if and only if G can be obtained from K_1 by a sequence of gated expansions.
- (iii) If G is a quasi-semimedian (resp. semimedian) graph then it can be obtained from K_1 by a sequence of (resp. binary) connected expansions.

4. Characterizing quasi-median graphs

For a relation R , let R^* stand for its transitive closure. We can prove straightforwardly that in quasi-semimedian graphs the relation \approx equals $(\delta \cup \kappa)^*$. Hence, this is also true for quasi-median graphs. The reverse implication need not be true in general. Nevertheless, these relations are important for the main result of this section:

Theorem 4.1. *The following assertions are equivalent for a connected graph G :*

- (i) G is a quasi-median graph.
- (ii) G is a partial Hamming graph with $\approx = (\delta \cup \kappa)^*$, and G has neither a Q_3^- nor a house as a convex subgraph.
- (iii) G is a quasi-semimedian graph, and G has neither a Q_3^- nor a house as a convex subgraph.

For the proof of this theorem we need a lemma. It states that sets W_i from Definition 3.3 enjoy the so-called *Helly property*. (It is well known that this property holds for gated subsets [29], hence the present lemma is seemingly a stronger variation of this result.)

Lemma 4.2. *Let G be a connected graph and let W_i , $i = 1, \dots, k$ be subsets of $V(G)$ which satisfy Definition 3.3. Then $\bigcap_{i=1}^k W_i \neq \emptyset$.*

Proof (Induction on k). The claim is true for $k = 2$. Suppose that the claim holds for $k \geq 2$, and let W_i , $i = 1, \dots, k+1$, be the subsets of $V(G')$ that satisfy the conditions in Definition 3.3. Observe that the sets W_i for $i = 1, \dots, k$ satisfy the conditions in Definition 3.3 also in a graph induced by $\bigcup_{i=1}^k W_i$ hence by induction $\bigcap_{i=1}^k W_i$ is nonempty. Set $U = \bigcap_{i=1}^k W_i$. Suppose that $U \cap W_{k+1} = \emptyset$, and let $x \in W_{k+1} \cap [\bigcup_{i=1}^k W_i]$ be a vertex as close to U as possible. Then there exist indices $j, \ell \in \{1, \dots, k\}$ such that $x \in [W_{k+1} \cap W_j] \setminus W_\ell$ and let y be a vertex of U closest to x . Since, by definition the subgraph induced by $W_{k+1} \cup W_\ell$ is isometric, it follows by Definition 3.3(3) that there exists a vertex $z \in W_{k+1} \cap W_\ell$ such that $z \in I(x, y)$. Hence, we have $d(x, y) = d(x, z) + d(z, y)$, thus z is closer to U than x , moreover $z \in W_{k+1} \cap [\bigcup_{i=1}^k W_i]$. This is a contradiction to the choice of x . \square

Proof (Of Theorem 4.1). For (i) \Rightarrow (ii) we only need to observe that a graph, having convex Q_3^- or a convex house, cannot be quasi-median.

(ii) \Rightarrow (iii): By Proposition 3.2 it is enough to prove that $\delta^* = \sim$, and we know already that $\delta^* \subseteq \sim$.

Let $ab \sim uv$. Using (ii) and the fact $\sim \subseteq \approx$, it follows that $ab(\delta \cup \kappa)^*uv$. Let $ab = x_0y_0, x_1y_1, x_2y_2, \dots, x_ky_k = uv$ be a sequence of edges such that $x_iy_i(\delta \cup \kappa)x_{i+1}y_{i+1}$ for $i = 0, \dots, k-1$. Assume that ab and uv are selected such that consecutive edges of the above sequence are in relation κ as few times as possible and, among such sequences, k is as small as possible. Clearly, if κ is not involved at all, we are done. Otherwise, by the minimality assumptions, ab and x_1y_1 are in the same clique, x_2y_2 is not in it, and x_1y_1 and x_2y_2 are opposite edges of an induced square.

Assume first that a, b, x_1 , and y_1 are pairwise different vertices. Then the vertices a, x_1, y_1, x_2, y_2 as well as b, x_1, y_1, x_2, y_2 induce houses, and as there is no convex house in G , any of these two houses gives a convex $K_3 \square K_2$. Let x'_1 , respectively y'_1 , be the vertices of the convex closure of the two houses. As G contains no $K_4 - e$ it follows that $x'_1 \neq y'_1$. By the same argument $x'_1y'_1$ is an edge which is the opposite edge of a square containing ab and lies in the same clique as x_2y_2 . As $ab \sim uv$ and $ab \sim x'_1y'_1$, transitivity implies $x'_1y'_1 \sim uv$. By minimality, $x'_1y'_1\delta^*uv$, and since $ab\delta x'_1y'_1$ we conclude that $ab\delta^*uv$.

Let now $a = x_1$ (and, of course, $b \neq y_1$). Then the vertices a, b, y_1, x_2, y_2 induce a house whose convex closure is $K_3 \square K_2$. Let x'_2 be the remaining vertex of the $K_3 \square K_2$. Then $x_2x'_2\kappa x_2y_2$ and so $x_2x'_2(\delta \cup \kappa)^*uv$. By the minimality we infer that $x_2x'_2\delta^*uv$ and as $ab\delta x_2x'_2$ we conclude again that $ab\delta^*uv$.

(iii) \Rightarrow (i): We will prove that G is a quasi-median graph by showing that G can be obtained by a sequence of gated expansions. From Theorem 3.4 (iii) we know that G can be obtained from K_1 by a sequence of connected expansions.

We first claim that for each expansion the sets W_i corresponding to it have the same pairwise intersections, i.e. $W_i \cap W_j = W_1 \cap W_2$ for all pairs of indices $1 \leq i < j \leq k$.

Note that this also means that the common intersection of all sets W_i is the same set, which we shall call U .

The claim is trivial for $k = 2$, so let $W_1, W_2, \dots, W_k, k \geq 3$, be the sets corresponding to the expansion. By Lemma 4.2 these sets have a common nonempty intersection. Suppose that $W_i \cap W_j \neq W_i \cap W_l$ for some indexes $i, j, l \in \{1, \dots, k\}$. Then, since the expansion is connected, there exists a vertex $x \in W_i \cap W_j \setminus W_l$ which is adjacent to a vertex $y \in W_i \cap W_j \cap W_l$. Let x_i, y_i, x_j, y_j, y_l be vertices of the graph G that is obtained from G' by this expansion, so that the indices of vertices and sets naturally corresponds. Obviously these vertices form a convex house G . Now, if this is not the last expansion in the sequence further expansions cannot change that we have a convex house in a graph. Indeed, this is obvious if the house lies entirely in one of the W_i 's of an expansion. If not, then by Definition 3.3(3) the intersection of two sets W_i, W_j in which the house is lying must include two vertices of the triangle of the house which are a cutset of the house. Clearly we also obtain the convex house in the graph obtained by this expansion. This contradiction proves the claim.

Now, let us assume that in one of the expansions of the sequence, the subgraph induced by $U = \bigcap_{i=1}^k W_i$ is not gated. Assume first that this happens in the last expansion step. Thus G' is quasi-median and G is obtained from G' by an expansion relative to the sets $W_i, i = 1, \dots, k$, having a common intersection U . Since G' is quasi-median it is a weakly modular graph, $\langle U \rangle$ is its triangle closed subgraph (this again follows from nonexistence of convex houses), therefore by Lemma 2.1, $\langle U \rangle$ is not weakly 2-convex. Thus there exist vertices $u, v \in U$ such that $d_{\langle U \rangle}(u, v) = 2$, and there is $x \in V(G) \setminus U$ which is a common neighbour of u and v . Let $w \in U$ be a common neighbour of u and v . Let $u', v', w', u'', v'', w''$ be vertices in G corresponding to vertices u, v , and w . (There can be more than two such triples, but we need just two.) Then $u', v', w', u'', v'', w''$ and x form a convex Q_3^- in G which is a contradiction. Similarly as above one can check that if this was not the last expansion step, further expansions cannot change that we have a convex Q_3^- in a graph. \square

From the above theorem we immediately obtain the following characterization of median graphs:

Corollary 4.3 ([4]). *A graph G is a median graph if and only if G is a semimedial graph that contains no convex Q_3^- .*

Proof. Use that median graphs are precisely bipartite quasi-median graphs, that bipartite partial Hamming graphs are precisely partial cubes, and that in bipartite graphs $\delta = \delta \cup \kappa$ and $\sim = \approx$. Then apply the first two assertions of Theorem 4.1. \square

To get another characterization of quasi-median graphs, we recall the following result.

Theorem 4.4 ([3]). *A connected graph G is a partial Hamming graph if and only if*

- (i) *the relation \approx is transitive,*
- (ii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and*
- (iii) *G has no isometric cycles C_{2n+1} for $n \geq 2$.*

Note that condition (iii) of the above theorem can be replaced by (cf. [3]):

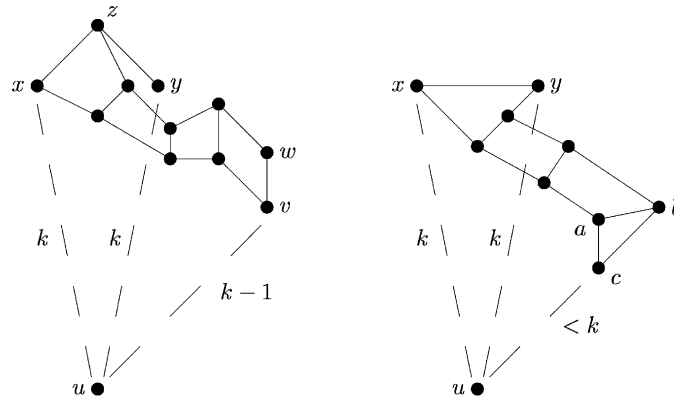


Fig. 2. Semi-quadrangle and semi-triangle property.

- (iii') If P is a path connecting the endpoints of an edge xy , then P contains an edge ab with $xy \approx ab$.

Combining Theorem 4.4 with 4.1 we get:

Corollary 4.5. A connected graph G is a quasi-median graph if and only if

- (i) $\approx = (\delta \cup \kappa)^*$,
- (ii) for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$,
- (iii) G has no isometric cycles C_{2n+1} for $n \geq 2$, and
- (iv) G has no convex Q_3^- and no convex house.

5. Characterizing quasi-semimedian graphs

In this section we examine quasi-semimedian graphs more closely. We extend a result from [18] by characterizing quasi-semimedian graphs among partial Hamming graphs. Then we prove a characterization of quasi-semimedian graphs which, together with Theorem 4.1, gives another characterization of quasi-median graphs. We begin with a new concept—a semi-quadrangle property. It generalizes the concept of the quadrangle property and will be used in a characterization of quasi-semimedian graphs.

A graph G satisfies the *semi-quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$ with z a common neighbour of x and y , there exists an edge wv such that $vw\delta^*xz$ and $d(u, v) = d(u, x) - 1$, cf. Fig. 2. (Note that in the definition of the quadrangle property a part of the condition that uses δ^* is changed to $vw\delta xz$ and $w = y$.)

For our next result we recall:

Lemma 5.1 ([8]). Let G be a partial Hamming graph, and K a clique in G . Then for any vertex $u \in V(G)$ the distances from u to vertices of K are either equal or there exists a unique $x \in K$ that is closer to u than other vertices of K .

Proposition 5.2. *A graph is quasi-semimedial if and only if it is a partial Hamming graph that satisfies the semi-quadrangle property.*

Proof. Let G be quasi-semimedial and let vertices u, x, y, z be as above. Let P_1 be a shortest path from x to u , and P_2 a shortest path from u to y . Without loss of generality, we may assume that u is the only common vertex of P_1 and P_2 . By Theorem 4.4 (iii') there exists an edge ab which lies on a path $x \rightarrow P_1 \rightarrow u \rightarrow P_2 \rightarrow y \rightarrow z$ and is in relation \approx with xz . Suppose that there exists a clique with edges e, f such that $xz \sim e$ and $f \sim ab$. Since ab is on a shortest path from u to z , one of the vertices a or b is closer to u than the other. Hence by Theorem 4.4 (ii) one of the endvertices of f is closer to u than the other, and by Lemma 5.1 we deduce that this endvertex of f is closer to u than both endvertices of e . This is a contradiction to $u \in W_{xz}$, since by Theorem 4.4 (ii) u should be closer to one endvertex of e . Thus, the remaining option is that $xz \sim ab$. Since G is quasi-semimedial, we derive by Proposition 3.2 that $xz\delta^*ab$, and the semi-quadrangle property now easily follows.

If G is not quasi-semimedial then by Proposition 3.2 there exist edges xy, uv such that $xy \sim uv$, but xy is not in relation δ^* with uv . In addition, we may choose xy and uv in such a way that the distances between their endvertices are as small as possible. Now, the semi-quadrangle property does not hold for vertices x, u , a neighbour of v which lies on a shortest path to y , and v . \square

Proposition 5.2 is analogous to the following characterization of median graphs from [21]: G is a median graph if and only if G is a partial cube satisfying the quadrangle property. Also, it implies the following characterization of semimedial graphs.

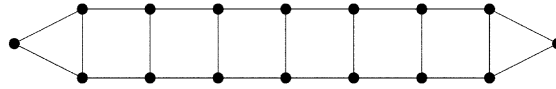
Corollary 5.3. *A graph is semimedial if and only if it is a partial cube that satisfies the semi-quadrangle property.*

We now introduce yet another concept—semi-triangle property. A graph G satisfies a *semi-triangle property* if for any vertices $u, x, y \in V(G)$ where $d(u, x) = d(u, y) = k \geq 2$ such that $xy \in E(G)$, there exists a triangle with vertices a, b, c such that $xy\delta^*ab$, and $d(u, a) - 1 = d(u, b) - 1 = d(u, c) < k$, cf. Fig. 2. (Note that in the definition of ordinary triangle property we have $a = x$ and $b = y$.) A graph is *semi-weakly-modular* if it satisfies both the semi-quadrangle and the semi-triangle property.

It is not hard to see that quasi-semimedial graphs are semi-weakly-modular. Indeed, let G be a quasi-semimedial graph, and vertices u, x, y as above. Let u' be the last vertex on a shortest path from u to x for which $d(u', x) = d(u', y)$. Thereby, there exist neighbours a, b of u' such that $a \in W_{xy}$ and $b \in W_{yx}$. Wilkeit showed:

Lemma 5.4 ([31]). *If G is a partial Hamming graph then: if a vertex $w \in V(G)$ has the same distance to adjacent vertices x and y of G , then any two neighbours $a \in W_{xy}$ and $b \in W_{yx}$ of w are adjacent.*

From this we infer that a, b and u' are in a triangle, and obviously $d(u, a) - 1 = d(u, b) - 1 = d(u, u') < d(u, x)$. Finally, by Proposition 3.2 it follows that $ab\delta^*xy$, so the semi-triangle property holds.

Fig. 3. Graph H_7 .

In the search for an analogue of [Theorem 3.1](#) we first observe that excluding graphs $K_4 - e$ and $K_{2,3}$ is not enough. For this sake consider graphs H_n obtained from grid graphs $P_2 \square P_n$ by attaching a triangle to each of both the edges with endvertices of degree 2, cf. [Fig. 3](#). The graphs H_n are semi-weakly-modular but not quasi-semimedian. Moreover, they are not even partial Hamming graphs.

Whenever H_n is an induced subgraph of a graph G we shall denote its vertices by $H_n(u, v)$ where u and v are the unique vertices of degree 2 in H_n . In the following theorem we prove that one must exclude graphs H_n as induced subgraphs for which $I_G(u, v) \cap H_n(u, v) \neq \{u, v\}$ holds. Also, instead of just excluding subgraphs $K_{2,3}$ we need a stronger condition taken from [Theorem 4.4](#) (ii). Note that this condition implies transitivity of the relation \sim , cf. [\[31\]](#).

Theorem 5.5. *A graph G is quasi-semimedian if and only if*

- (i) *G is semi-weakly-modular;*
- (ii) *for every induced H_n , $n \geq 1$, we have $I_G(u, v) \cap H_n(u, v) = \{u, v\}$,*
- (iii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$.*

Proof. By the above discussion we only need to prove that conditions are sufficient, and by [Proposition 5.2](#) it is enough to show that conditions (i)–(iii) ensure that G is a partial Hamming graph. Moreover, by [Theorem 4.4](#) we only need to prove conditions (i) and (iii) of that theorem.

First we claim that the condition of [Lemma 5.1](#) holds for G . (In the proof we shall recall that $\delta^* \subseteq \sim$, since \sim is transitive.) Assume that there is a clique with vertices x, y, z , and a vertex $u \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$. Let u be a vertex closest to z with this property, hence the neighbour x' on a shortest path from u to x is in W_{xy} . By the semi-triangle property there exists a triangle with vertices a, b, c such that $ab\delta^*xy$, and $d(u, c) = d(u, a) - 1 = d(u, b) - 1$. Note that, by condition (iii) of the theorem, since $x' \in W_{xy}$, also $x' \in W_{ab}$. It is clear that we have an induced subgraph H_n with vertex set $H_n(c, z)$, hence $I_G(c, z) \cap H_n(c, z) = \{c, z\}$. Thus $z \in W_{ca}$, and again by (iii) we infer that $z \in W_{ux'}$ (because $ux' \sim ca$), which is a contradiction.

Secondly, we prove that the condition of [Lemma 5.4](#) holds for G . Let $w \in V(G)$ be a vertex having the same distance to adjacent vertices x and y of G , and let $u \in W_{xy}$ and $v \in W_{yx}$ be the neighbours of w . By the semi-triangle property there exists a triangle with vertices a, b, c such that $ab\delta^*xy$, and $d(w, c) = d(w, a) - 1 = d(w, b) - 1$. By condition (iii) of the theorem we have $u \in W_{ab}$, $v \in W_{ba}$, and by the claim of the previous paragraph $u \in W_{ac}$. Hence, again using (iii), we deduce $d(u, v) = d(u, w) = 1$ as claimed.

We next prove condition (i) of [Theorem 4.4](#). Suppose not: then there exist edges e, f , and g such that $e \approx f$ and $f \approx g$ but e and g are not in relation \approx . Now, in both cases where relation \approx holds, it is clear that it is not equal to \sim . Hence there exist cliques

C and C' , and edges e' of C and g' of C' , such that $f \sim e'$ and $f \sim g'$. Now, it is not hard to prove that $f\delta^*e'$ and $f\delta^*g'$. (Use the semi-quadrangle property in the cycle formed by f , e' and shortest paths between their endvertices, and then use the induction on the distance between endvertices of two edges in relation \sim .) Thus we have at least one H_n in G , moreover, using a shortest δ^* sequence we can choose a H_n which is induced. (The number of induced H_n 's in G depends on the sizes of C and C' .) By the condition of Lemma 5.1 each vertex of C is either closest to exactly one of the vertices of C' , or is at the same distance to all of them. If the latter holds for a vertex z of C , then obviously $H_n(z, w) \cap I(z, w) \neq \{z, w\}$ for any w of C' , and we are through in this case. On the other hand, if all vertices of C and C' have their unique closest vertices, then we deduce that e' and g' are in relation \sim , hence $e \approx g$, and so \approx is transitive in G .

Finally, we prove the condition (iii) of Theorem 4.4. Suppose that the odd cycle $C : x \rightarrow z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_{2k+1} = x$ is isometric. Then the condition of Lemma 5.4 can be used for x, z_k, z_{k+1} , and z_1, z_{2k} , which says that z_1 and z_{2k} are adjacent. This proves that G is a partial Hamming graph, and thus quasi-semimedial. \square

Combining Theorem 5.5 with 4.1 we obtain yet another characterization of quasi-medial graphs:

Corollary 5.6. *A connected graph G is a quasi-medial graph if and only if*

- (i) G is semi-weakly-modular;
- (ii) G has no induced H_n , $n \geq 1$, for which $I_G(u, v) \cap G_n(u, v) \neq \{u, v\}$,
- (iii) for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and
- (iv) G has no convex Q_3^- and no convex house.

6. Tree-like equalities for quasi-medial graphs

Median graphs simultaneously generalize trees and hypercubes. Moreover, they are considered to be the class which reflects all important properties shared by these two classes (see Mulder's metaconjecture [26]). Soltan and Chepoi [28] and Škrekovski [27] proved tree-like equalities for median graphs which shed a surprising light on the metaconjecture. Indeed, let q_r be the number of subgraphs of a median graph isomorphic to Q_r , and let k be the number of its equivalence classes with respect to the relation \sim . Then

$$\sum_{i \geq 0} (-1)^i q_i = 1 \quad \text{and} \quad k = - \sum_{i \geq 0} (-1)^i i q_i.$$

Note that the second one applied to trees tells that the number of equivalence classes with respect to the relation \sim equals the number of edges—a less known characterizing property of trees. These relations also imply the Euler-type formulae from [22, 23], and they were widely generalized in [5]. In the following result we will extend the above equalities to the quasi-medial graphs by using subgraphs which are isomorphic to Hamming graphs.

(We note here that invariant (with respect to automorphisms) Hamming subgraphs of quasi-median graphs were studied by Chastand and Polat [7].)

For a Hamming graph $H = K_{k_1} \square K_{k_2} \square \cdots \square K_{k_n}$, with $k_i > 1$ for all i , we say that n is the *dimension* of H . The *dimension* of a partial Hamming graph G is the dimension of a Hamming graph of smallest dimension into which G can be isometrically embedded. Alternatively, the dimension of G is the number of expansion steps with which G is obtained from K_1 , which in turn coincides with the number of \approx classes in G (using Lemma 4.2). Note that the dimension of Q_n is n .

Theorem 6.1. *Let G be a quasi-median graph of dimension t and let α_i ($i \geq 0$) be the number of induced Hamming subgraphs of G of degree i . Then*

$$\sum_{i \geq 0} (-1)^i \alpha_i = 1 \quad \text{and} \quad -t = \sum_{i \geq 0} (-1)^i i \alpha_i.$$

Proof. The proof is by induction on the number of vertices. The claim is obviously true for $G \cong K_1$. So, we may assume that G is constructed by a gated expansion from a quasi-median graph G' with respect to U, W_1, \dots, W_n . Let α_i^0 (resp. α'_i) be the number of induced subgraphs of $\langle U \rangle$ (resp. G') isomorphic to some Hamming graph of degree i . Denote by t_0 and t' the dimensions of $\langle U \rangle$ and G' , respectively. Since G' and $\langle U \rangle$ are quasi-median graphs, by induction, we assume that the above two relations are valid for these two graphs. It is not hard to observe that

$$\alpha_k = \alpha'_k - \alpha_k^0 + \sum_{i \geq 0} \alpha_{k-i}^0 \binom{n}{i+1}.$$

Recall that $\binom{n}{i} = 0$ whenever $i > n$. In what follows, we will use the following two identities:

$$\sum_{i \geq 1} (-1)^i \binom{n}{i} = -1 \quad \text{and} \quad \sum_{i \geq 0} (-1)^i i \binom{n}{i} = 0.$$

We can now derive

$$\begin{aligned} \sum_{k \geq 0} (-1)^k \alpha_k &= \sum_{k \geq 0} (-1)^k \alpha'_k - \sum_{k \geq 0} (-1)^k \alpha_k^0 + \sum_{k \geq 0} (-1)^k \sum_{i \geq 0} \alpha_{k-i}^0 \binom{n}{i+1} \\ &= 1 - 1 + \sum_{k \geq 0} \left((-1)^{k-1} \alpha_k^0 \sum_{j \geq 1} (-1)^j \binom{n}{j} \right) \\ &= \sum_{k \geq 0} (-1)^k \alpha_k^0 = 1. \end{aligned}$$

Observe that $t = t' + 1$. For the second equality,

$$\begin{aligned}
 \sum_{k \geq 0} (-1)^k k \alpha_k &= \sum_{k \geq 0} (-1)^k k \alpha'_k - \sum_{k \geq 0} (-1)^k k \alpha_k^0 \\
 &\quad + \sum_{k \geq 0} \left((-1)^k k \sum_{i \geq 0} \alpha_{k-i}^0 \binom{n}{i+1} \right) \\
 &= -t' + t_0 + \sum_{k \geq 0} \left((-1)^{k-1} \alpha_k^0 \left(\sum_{j \geq 0} (-1)^{j+1} (k+j) \binom{n}{j+1} \right) \right) \\
 &= -t' + t_0 + \sum_{k \geq 0} \left(\alpha_k^0 \sum_{j \geq 0} (-1)^{k+j} (k+j) \binom{n}{j+1} \right) \\
 &= -t' + t_0 + \sum_{k \geq 0} \left((-1)^{k-1} (k-1) \alpha_k^0 \sum_{j \geq 0} (-1)^{j+1} \binom{n}{j+1} \right) \\
 &\quad + \sum_{k \geq 0} \left((-1)^k \alpha_k^0 \sum_{j \geq 0} (-1)^{j+1} (j+1) \binom{n}{j+1} \right) \\
 &= -t' + t_0 + \sum_{k \geq 0} (-1)^k (k-1) \alpha_k^0 + \sum_{k \geq 0} (-1)^k \alpha_k^0 0 \\
 &= -t' + t_0 + \sum_{k \geq 0} (-1)^k (k-1) \alpha_k^0 \\
 &= -t' + t_0 + \sum_{k \geq 0} (-1)^k k \alpha_k^0 - \sum_{k \geq 0} (-1)^k \alpha_k^0 \\
 &= -t' + t_0 - t_0 - 1 = -t. \quad \square
 \end{aligned}$$

The equalities of [Theorem 6.1](#) cannot be extended to quasi-semimedial graphs, not even in the bipartite case. However, these relations imply an Euler-type formula which can be extended to a larger class of graphs. We are going to prove it for graphs that can be obtained by a connected expansion procedure. Note that these graphs include the class of quasi-semimedial graphs, and that this result extends all such previously known formulae [4, 22].

Theorem 6.2. *Let G be a graph with n vertices, m edges and of dimension k , that is obtained by a sequence of connected expansions from K_1 . Then $2n - m - k \leq 2$. Moreover equality holds if and only if G is $C_t \square K_2$ -free ($t \geq 3$) and K_4 -free.*

Proof. The proof is by induction on k . Let G be obtained from a graph G' by a connected expansion with respect to W_1, W_2, \dots, W_r . Let $W^* = \bigcup_{1 \leq i < j \leq r} (W_i \cap W_j)$. For $i = 1, \dots, r$ denote by a_i and b_i the number of vertices and edges, respectively, that lie in at least i covering subsets of G' . Let k' be the dimension of G' , then $k = k' + 1$.

Clearly, a_1 and b_1 are the number of vertices and edges, respectively, of G' . Moreover, $\sum_{i=2}^r b_i$ is the number of edges added by the expansion to the \approx classes of G' , while $\sum_{i=1}^k (i-1)a_i$ is the number of edges of the new \approx class. Hence,

$$n = \sum_{i=1}^r a_i \quad \text{and} \quad m = \sum_{i=1}^r (b_i + (i-1)a_i),$$

from which we obtain

$$\begin{aligned} 2n - m - k &= \sum_{i=1}^r (3-i)a_i - \sum_{i=1}^r b_i - (k' + 1) \\ &= (2a_1 - b_1 - k') + \sum_{i=2}^r (3-i)a_i - \sum_{i=2}^r b_i - 1. \end{aligned}$$

By the induction hypothesis, $2a_1 - b_1 - k' \leq 2$ holds for G' , therefore

$$\begin{aligned} 2n - m - k &\leq \sum_{i=2}^r (3-i)a_i - \sum_{i=2}^r b_i + 1 \\ &= (a_2 - b_2 + 1) + \sum_{i=3}^r (3-i)a_i - \sum_{i=3}^r b_i \\ &\leq a_2 - b_2 + 1. \end{aligned}$$

Now, a_2 and b_2 are the numbers of vertices and edges of $\langle W^* \rangle$, respectively. Since $\langle W^* \rangle$ is connected, we have $a_2 - b_2 \leq 1$, which proves the theorem's inequality.

For the second part of the theorem observe that the equality will hold precisely when G is obtained by an expansion procedure in such a way that in all the expansions the numbers a_i , $i \geq 4$, and b_j , $j \geq 3$, are zero and $a_2 - b_2 = 1$. This holds precisely when in each expansion step at most three covering sets are involved, no edge lies in all three covering sets (which means that their common intersection is a vertex), and $\langle W^* \rangle$ is a tree. Obviously no K_4 , $C_3 \square K_2$, $C_4 \square K_2$, $C_5 \square K_2$ can then appear in G , and it is straightforward to check by induction on the dimension, that this holds also for $C_t \square K_2$ ($t \geq 6$). The converse is obvious. \square

Corollary 6.3. *Let G be a planar graph with n vertices and of dimension k , that is obtained by a sequence of connected expansions from K_1 . Let f be the number of faces in its planar embedding. Then $f \geq n - k$, where equality holds if and only if G is $C_t \square K_2$ -free ($t \geq 3$) and K_4 -free.*

Proof. Combine Theorem 6.2 with Euler's formula $n - m + f = 2$. \square

Acknowledgement

This work was supported by the Ministry of Education, Science and Sport of Slovenia under the grants Z1-3073-0101-01, 0101-P-504, and Z1-3129-0101-01, respectively.

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